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COMPUTER-EXTENDED SERIES FOR A SOURCE/SINK DRIVEN GAS CENTRIFUGE*

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SUMMARY

We have reformulated the general problem of internal flow in a modern, high speed gas centrifuge with sources and sinks in such a way as to obtain new, simple, rigorous closed form analytical solutions. Both symmetric and antisymmetric drives lead us to an ordinary differential equation in place of the usual inhomogeneous Onsager partial differential equation. Owing to the difficulties of exactly solving this sixth order, inhomogeneous, variable coefficient ordinary differential equation we appeal to the power of perturbation theory and techniques. Two extreme parameter regimes are identified, the so-called semi-long bowl approximation and a new short bowl approximation. Only the former class of problems is treated here. The long bowl solution for axial drive is the correct leading order term, just as for pure thermal drive. New $O(1)$ results are derived for radial, drag and heat drives in two dimensions. Then regular asymptotic, even ordered power series expansions for the flow field are carried out on the computer to $O(\varepsilon^4)$ using MACSYMA. These approximations are valid for values of ε near unity. In the spirit of Van Dyke, one can carry out this expansion process, in theory, to apparently arbitrary order for arbitrary but finite decay length ratio. Curiously, the flows induced by axial and radial forces are proportional for asymptotically large source scale heights, x^*. Corresponding isotope separation integral parameters will be given in a companion paper.

KEY WORDS Computer Extended Series Rotating Fluid Mechanics MACSYMA

INTRODUCTION

Consider the general problem of approximate two dimensional solutions to Onsager's pancake equation with sources and sinks and compressible Ekman boundary layers on horizontal rotating surfaces. Our discussion of computer extended series for this special equation continues here. The reader is presumed to be familiar with it.[1] Briefly, the axisymmetric flow field inside a gas filled, rotating, circular cylinder (i.e. a gas centrifuge) is governed by the linearized Navier–Stokes and energy equations for small perturbations about classical rigid-body rotation. Under the appropriate set of approximations the system of eight simultaneous partial differential equations can be represented by the well-known linear partial differential equation due first and foremost to Onsager, and which bears his name,[2]

$$L\tilde{\chi} = L_6\tilde{\chi} + B^2\tilde{\chi}_{yy} = \tilde{F}(x, y), \tag{1}$$

where $\tilde{\chi}$ is the master potential and

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$$L_6\tilde{\chi} = [e^x(e^x\tilde{\chi}_{xx})_{xx}]_{xx}.$$

Equation (1) is subject to the nine boundary conditions,

$$\begin{aligned}
&\tilde{\chi}_x(0,y) = \tilde{\chi}_{xx}(0,y) = 0, \quad L_5\tilde{\chi}(0,y) = \frac{Re}{32A^{10}}\tilde{\theta}_y(y),\\
&\tilde{\chi}(\infty,y) = \tilde{\chi}_y(\infty,y) = \tilde{\chi}_x(\infty,y) = L_3\tilde{\chi}(\infty,y) = 0,\\
&\tilde{\chi}_y(x,0) = -4S^{-1/4}Re^{-1/2}A^4[e^{x/2}\tilde{\chi}_x(x,0)]_x,\\
&\tilde{\chi}_y(x,y_0) = 4S^{-1/4}Re^{-1/2}A^4[e^{x/2}\tilde{\chi}_x(x,y_0)]_x,
\end{aligned} \tag{2}$$

where

$$L_5\tilde{\chi} = [e^x(L_3\tilde{\chi})_x]_x, \quad L_3\tilde{\chi} = (e^x\tilde{\chi}_{xx})_x$$

and the stream function and mass velocity are given by

$$\tilde{\psi} = -2A^2\tilde{\chi}_x, \quad \rho_0\tilde{w} = 4A^4\tilde{\chi}_{xx}.$$

Flow field variables are proportional to certain partial derivatives of the master potential function. The inhomogeneity due to internal source/sinks, such as feed or a (withdrawal) scoop, is[3]

$$\tilde{F}(x,y) = \frac{B^2A^2}{2ReS}\left\{\int_x^\infty(\tilde{T}_y - 2\tilde{V}_y)\,dx' - [(e^x\tilde{U}_y)_x + (e^x\tilde{W})_{xx}]\right\}. \tag{3}$$

The mass source/sink term is neglected, and the superscript tilde denotes a multivariate function. Here $\tilde{U}$, $\tilde{V}$, $\tilde{W}$, $\tilde{T}$ are source/sinks of radial, azimuthal and axial momenta and heat, respectively. In a companion paper the corresponding isotope separation parameters, E and M, will be reported. Finite element approximations have been given for concentrated point sources and sinks along with approximate 'local' analytical solutions for the singularities.[4] But no one has previously reported 'global' solutions for the class of internally driven problems in terms of simple, classical mathematical functions. A catalogue of high order results is presented.

PERTURBATION ANALYSIS: SEMI-LONG BOWL APPROXIMATION

Assume simple harmonic expressions for both antisymmetric and symmetric drive mechanisms:

$$\tilde{F}(x,y) = F(x)\sin(N\pi y/y_0), \quad \tilde{\chi}(x,y) = \chi(x)\sin(N\pi y/y_0), \quad N = 1,2,3,\ldots \tag{4}$$

and

$$(\tilde{U},\tilde{V},\tilde{T}) = (U_0,V_0,T_0)\,\delta(x-x^*)\cos(N\pi y/y_0), \quad \tilde{W} = W_0\delta(x-x^*)\sin(N\pi y/y_0), \tag{5}$$

where U_0, V_0, W_0, T_0 are source/sink strengths, of course. Equation (4) requires replacing the end Ekman boundary layers in equation (2) with free slip, impermeable boundary conditions. Approximations for the Ekman suction will be derived in a separate paper. Combining expressions (3), (4) and (5) gives

$$F(x) = c(N\pi/y_0)\left\{(T_0 - 2V_0)\,\mathsf{U}(x^* - x) - [U_0(e^x\delta(x-x^*))_x + \frac{W_0}{N\pi/y_0}(e^x\delta(x-x^*))_{xx}]\right\}. \tag{6}$$

Here $c \equiv -B^2A^2/(2ReS) = -8/A^4$ and $\mathsf{U}(x-x^*)$ is the Heaviside unit step function. Now combining (1) and (4) it is possible to write

$$L_6\chi(x) - E^2\chi(x) = F(x), \tag{7}$$

under appropriate boundary conditions, where $E = BN\pi/y_0$.[1]

Defining $\varepsilon = E \ll 1$ produces a differential problem in a small parameter, and assuming an even ordered power series expansion, substituting into the ordinary differential equation (7) and collecting like ordered terms gives

$$L_6\chi_M = \mathrm{U}(0^+ - M)F(x) + \chi_{M-2}, \quad M = 0, 2, \ldots, \quad \chi_{-2} \equiv 0, \tag{8}$$

also subject to the appropriate boundary conditions. Now, all that is needed to close the $O(\varepsilon^M)$ problem is the one-dimensional Green's function $G(x, x^*)$ of L_6.[5]

$$\begin{aligned} G_{x^*>x} &= \tfrac{-1}{2}(\mathrm{e}^{-2x}\mathrm{e}^{-x^*} + \mathrm{e}^{-x}\mathrm{e}^{-2x^*}) + \tfrac{1}{8}\mathrm{e}^{-2x}\mathrm{e}^{-2x^*} - x\mathrm{e}^{-x}\mathrm{e}^{-x^*} + \tfrac{1}{4}(x^* - x + 3)\mathrm{e}^{-2x^*}, \quad x^* > x,\\ G_{x^*<x} &= \tfrac{-1}{2}(\mathrm{e}^{-2x}\mathrm{e}^{-x^*} + \mathrm{e}^{-x}\mathrm{e}^{-2x^*}) + \tfrac{1}{8}\mathrm{e}^{-2x}\mathrm{e}^{-2x^*} - x^*\mathrm{e}^{-x}\mathrm{e}^{-x^*} + \tfrac{1}{4}(x - x^* + 3)\mathrm{e}^{-2x}, \quad x^* < x. \end{aligned} \tag{9}$$

The aforementioned boundary conditions on χ_M (or for that matter χ) are included in $G(x, x^*)$. To be brief, formulae for χ_M and simple asymptotic results for χ_M'' are given along with illustrations of only χ_∞''.

$O(1)$ SOLUTIONS

Pure axial drive

The leading order or basic (finite bowl) solution is simply the long bowl solution for pure axial drive multiplied by $\sin(N\pi y/y_0)$. For reference, the less-than-well-known axial drive long bowl result is repeated here.[6]

$$\begin{aligned} \chi_{0_<}(x) &= cW_0[2(1 - \mathrm{e}^{-x^*})(\mathrm{e}^{-x} - \tfrac{1}{4}\mathrm{e}^{-2x}) - (x + 2)\mathrm{e}^{-x} + (x^* - x + 2)\mathrm{e}^{-x^*}], \quad x < x^*,\\ \chi_{0_>}(x) &= cW_0[2(1 - \mathrm{e}^{-x^*})(\mathrm{e}^{-x} - \tfrac{1}{4}\mathrm{e}^{-2x}) - x^*\mathrm{e}^{-x}], \quad x > x^*. \end{aligned} \tag{10}$$

Heat and drag drive

$$L_6\chi_0 = c(N\pi/y_0)(T_0 - 2V_0)\mathrm{U}(x^* - x). \tag{11}$$

Using the powerful 'solving' property of Green's functions,

$$\begin{aligned} \chi_0(x) &= c(N\pi/y_0)(T_0 - 2V_0)\int_0^\infty G(x;\xi)\mathrm{U}(x^* - \xi)\,\mathrm{d}\xi\\ &= c(N\pi/y_0)(T_0 - 2V_0)\int_0^{x^*} G(x;\xi)\,\mathrm{d}\xi. \end{aligned} \tag{12}$$

If $x < x^*$

$$\chi_0(x) = c(N\pi/y_0)(T_0 - 2V_0)\left[\int_0^x G_< + \int_x^{x^*} G_>\right],$$

and if $x > x^*$

$$\chi_0(x) = c(N\pi/y_0)(T_0 - 2V_0)\int_0^{x^*} G_<. \tag{13}$$

Since both the Green's function and the source/sink problems have the same homogeneous boundary conditions, we do not have to deal with them. As before, the required exact integrations are obtained from the symbolic algebraic manipulation code MACSYMA.[7] We find

$$\chi_{0_<} = -\pi c N\frac{T_0-2V_0}{16y_0}\{e^{-2x^*-4x}[((16x-7)e^{2x}+12e^x-1)e^{2x^*}-(16xe^{3x}+8e^{2x})e^{x^*}$$
$$+2e^{4x}x^*+(7-2x)e^{4x}-4e^{3x}+e^{2x}]+e^{-4x}(20e^{3x}+(-2x^2-28x-9)e^{2x}-12e^x+1)\},$$

$$\chi_{0_>} = -\pi c N\frac{T_0-2V_0}{16y_0}e^{-2x^*-2x}[(2x^{*2}-(4x+12)x^*+20e^x+7)e^{2x^*}$$
$$-(16e^x x^*+16e^x+8)e^{x^*}-4e^x+1]. \tag{14}$$

For asymptotically large x^*, this reduces to simply

$$\chi''_{0_\infty} = \lim_{x^*\to\infty}\chi''_{0_<} = cN\pi(T_0-2V_0)e^{-x}[-5+(2x^2+8x+5)e^{-x}], \tag{15}$$

for the non-dimensional asymptotic axial mass velocity divided by $4A^4$.

Pure radial drive

$$L_6\chi_0 = -c(N\pi/y_0)U_0(e^x\delta(x-x^*))_x. \tag{16}$$

So

$$\chi_0 = -c(N\pi/y_0)U_0\int_0^\infty G(x;\xi)[e^\xi\delta(\xi-x^*)]_\xi\,d\xi. \tag{17}$$

Integrating by parts, and assuming $x^*\neq 0$, ∞ we have simply

$$\chi_0 = \begin{cases} c(N\pi/y_0)U_0e^{x^*}G_{x^*>}(x;x^*), \\ c(N\pi/y_0)U_0e^{x^*}G_{x^*<}(x;x^*). \end{cases} \tag{18}$$

$$\chi_{0_<} = \frac{cN\pi U_0}{4y_0}e^{-x^*-2x}[(4xe^x+2)e^{x^*}-2x^*e^{2x}+(2x-5)e^{2x}+4e^x-1],$$
$$\chi_{0_>} = -\frac{cN\pi U_0}{4y_0}e^{-x^*-2x}[e^{2x^*}+(-4x^*e^x+4e^x-2)e^{x^*}-4e^x+1]. \tag{19}$$

This reduces to just

$$\chi''_{0_\infty} = \frac{cN\pi U_0}{y_0}e^{-x}[(x-2)+2e^{-x}], \tag{20}$$

for asymptotically large x^*. A similar result was found earlier for an asymptotic axial drive (i.e. limit of equation (10)).[6] So, the same kind of flow is produced by application high in the atmosphere of a radial or an axial force distribution, although the magnitude of the flow induced by a unit radial force is much less than that for a corresponding unit axial force.

$O(\varepsilon^2)$ SOLUTIONS

Using the 'solving' property again gives

$$\chi_2(x) = \int_0^\infty G(x;\xi)\chi_0(\xi)\,d\xi, \tag{21}$$

and straightforward integration yields

$$\chi_2(x) = \chi_{2_{U_0}} + \chi_{2_{T_0-2V_0}} + \chi_{2_{W_0}}. \tag{22}$$

As you might expect formulae for finite, non-zero ε are much more complicated than the basic solutions.

Axial drive

For $x < x^*$

$$\chi_{2_<} = \int_0^\infty G(x;\xi)\chi_0(\xi;x^*)\,d\xi$$
$$= \int_0^x G_<\chi_{0_<} + \int_x^{x^*} G_>\chi_{0_<} + \int_{x^*}^\infty G_>\chi_{0_>}.$$

For $x > x^*$

$$\chi_{2_>} = \int_0^\infty G(x;\xi)\chi_0(\xi;x^*)\,d\xi$$
$$= \int_0^x G_<\chi_{0_<} + \int_x^{x^*} G_<\chi_{0_>} + \int_{x^*}^\infty G_>\chi_{0_>}. \tag{23}$$

Inspection of results for χ'' indicates that the largest numerical constant in the $O(\varepsilon^2)$ terms is 2 and it appears that the series is convergent to at least second order.

$$\chi_< = cW_0\Bigg\{\varepsilon^2\Bigg[\frac{e^{-x^*-6x}}{3456}(((1728x^2-96x-352)e^{3x}+(2016x+273)e^{2x}+(456-144x)e^x-54)e^{x^*}$$
$$+((1512-3456x)e^{4x}-2592e^{3x}+216e^{2x})x^*+(3456x^2-4968x+2160)e^{4x}$$
$$+(6048x-4944)e^{3x}+(3555-792x)e^{2x}-792e^x+54)$$
$$-\frac{e^{-5x^*-2x}}{3456}((((1728xe^x+864)x^*-2592xe^x-1296)e^{3x^*}+(-288e^{2x}x^{*2}+((288x-408)e^{2x}$$
$$+576e^x-144)x^*+(1808-552x)e^{2x}+(4032x-1104)e^x+2292)e^{2x^*}+(-684e^{2x}x^*$$
$$+(684x-2271)e^{2x}+(1368-576x)e^x-630)e^{x^*}+108e^{2x}x^*+(351-108x)e^{2x}-216e^x+54)$$
$$+\frac{e^{-5x^*-2x}}{1152}((((576xe^x+288)x^*-1152xe^x-576)e^{3x^*}+(-96e^{2x}x^{*2}+((96x-128)e^{2x}$$
$$+192e^x-48)x^*+(640-192x)e^{2x}+(1344x-384)e^x+768)e^{2x^*}+(-228e^{2x}x^*$$
$$+(228x-757)e^{2x}+(456-192x)e^x-210)e^{x^*}+36e^{2x}x^*+(117-36x)e^{2x}-72e^x+18)$$
$$+\frac{e^{-x^*-2x}}{576}((244e^x-180x-131)e^{x^*}+(-720e^x-252)x^*+196e^x-252x-519)$$
$$-\frac{e^{-x^*-6x}}{576}((288x^2e^{3x}+(336x+46)e^{2x}+(76-24x)e^x-9)e^{x^*}+((-72x^2-1008x-576)e^{4x}$$
$$-432e^{3x}+36e^{2x})x^*+(24x^3+648x^2-864x)e^{4x}+(1008x-792)e^{3x}+(592-132x)e^{2x}$$
$$-132e^x+9)\Bigg]+(x^*-x+2)e^{-x^*}+2\left(e^{-x^*}-\frac{e^{-2x}}{4}\right)(1-e^{-x^*})-(x+2)e^{-x}\Bigg\},$$

$$\chi_> = cW_0\Bigg\{\varepsilon^2\Bigg[\frac{e^{-x^*-2x}}{576}((244e^x - 180x - 131)e^{x^*} - (720e^x + 252)x^* + 196e^x - 252x - 519)$$

$$-\frac{e^{-5x^*-2x}}{576}((24x^{*3} + (72 - 72x)x^{*2} - (432x + 1008)x^* - 144x - 144)e^{4x^*}$$

$$+ (288e^x x^{*2} + (468 - 288e^x)x^* + 144e^x - 324x - 882)e^{3x^*} + ((768e^x - 60)x^*$$

$$+ 136e^x + 36x + 472)e^{2x^*} + (-96e^x x^* + 196e^x - 105)e^{x^*} - 36e^x + 9)$$

$$+\frac{e^{-5x^*-2x}}{576}((144x^{*2} - (144x + 576)x^* + 288x + 576)e^{4x^*} + (288e^x x^{*2} + (468 - 432e^x)x^*$$

$$- 288e^x - 324x - 954)e^{3x^*} + ((768e^x - 60)x^* + 128e^x + 36x + 474)e^{2x^*}$$

$$+ (-96e^x x^* + 196e^x - 105)e^{x^*} - 36e^x + 9)$$

$$-\frac{e^{-x^*-6x}}{576}((((288x - 144)e^{3x} + 240e^{2x} - 24e^x)x^* + (288 - 576x)e^{3x}$$

$$+ (96x - 538)e^{2x} + 132e^x - 9)e^{x^*} + (576x - 288)e^{3x} + (538 - 96x)e^{2x} - 132e^x + 9)$$

$$+\frac{e^{-x^*-6x}}{1152}((((576x - 320)e^{3x} + 480e^{2x} - 48e^x)x^* + (640 - 1152x)e^{3x} + (192x - 1077)e^{2x}$$

$$+ 264e^x - 18)e^{x^*} + (1152x - 640)e^{3x} + (1077 - 192x)e^{2x} - 264e^x + 18)\Bigg]$$

$$+ 2(e^{-x} - \tfrac{1}{4}e^{-2x})(1 - e^{-x^*}) - e^{-x}x^*\Bigg\}. \tag{24}$$

Heat and drag drive

$$\chi_< = \pi cN(T_0 - 2V_0)\Bigg\{\varepsilon^2\Bigg[\frac{e^{2x^*-6x}}{110{,}592}(((69{,}120x - 38{,}400)e^{3x} + (-4608x^3 - 27{,}912x^2 - 28{,}744x$$

$$+ 85{,}005)e^{2x} - (4032x^2 + 26{,}592x + 45{,}944)e^x + 432x^2 + 2808x + 4158)e^{2x^*}$$

$$+ ((-55{,}296x^2 + 3072x + 11{,}264)e^{3x} - (64{,}512x + 8736)e^{2x} + (4608x - 14{,}592)e^x$$

$$+ 1728)e^{x^*} + ((13{,}824x - 6048)e^{4x} + 10{,}368e^{3x} - 864e^{2x})x^* + (-13{,}824x^2$$

$$+ 40{,}608x - 17{,}712)e^{4x} + (35{,}328 - 24{,}192x)e^{3x} + (3168x - 15{,}516)e^{2x} + 3168e^x - 216)$$

$$-\frac{e^{-6x^*-2x}}{110{,}592y_0}((69{,}120xe^x + 34{,}560)e^{4x^*} + ((-4608xe^x - 2304)x^{*2} + (-11{,}520e^{2x}$$

$$- 86{,}016xe^x - 43{,}008)x^* + (11{,}520x - 38{,}400)e^{2x} + (23{,}040 - 40{,}192x)e^x - 25{,}856)e^{3x^*}$$

$$+ (864e^{2x}x^{*3} + ((17{,}640 - 864x)e^{2x} - 1728e^x + 432)x^{*2} + ((55{,}284 - 14{,}832x)e^{2x} - 29{,}664e^x$$

$$+ 7416)x^* + (20{,}957 - 6204x)e^{2x} - (32{,}256x + 12{,}408)e^x - 13{,}026)e^{2x^*} + (5760e^{2x}x^*$$

$$+ (18{,}912 - 5760x)e^{2x} + (2304x - 11{,}520)e^x + 4032)e^{x^*} - 432e^{2x}x^*$$

$$+ (432x - 1404)e^{2x} + 864e^x - 216)$$

$$
\begin{aligned}
&+\frac{e^{-6x^*-2x}}{9216y_0}((5760xe^x+2880)e^{4x^*}+((-384xe^x-192)x^{*2}-(960e^{2x}+7168xe^x+3584)x^*\\
&+(960x-3200)e^{2x}+(1920-3264x)e^x-2112)e^{3x^*}+(72e^{2x}x^{*3}+((1470-72x)e^{2x}\\
&-144e^x+36)x^{*2}+((4606-1236x)e^{2x}-2472e^x+618)x^*+(1741-516x)e^{2x}\\
&-(2688x+1032)e^x-1086)e^{2x^*}+(480e^{2x}x^*+(1576-480x)e^{2x}+(192x-960)e^x+336)e^{x^*}\\
&-36e^{2x}x^*+(36x-117)e^{2x}+72e^x-18)\\
&+\frac{e^{-2x^*-2x}}{55{,}296}((10{,}756e^x-7344x-6745)e^{2x^*}+(-23{,}424e^x+17{,}280x+12{,}576)e^{x^*}\\
&+(8640e^x+3024)x^*+10{,}608e^x+3024x+10{,}764)\\
&-\frac{e^{-2x^*-6x}}{55{,}296y_0}(((34{,}560x-17{,}280)e^{3x}-(2304x^3+13{,}968x^2+14{,}496x-42{,}184)e^{2x}\\
&+(-2016x^2-13{,}296x-22{,}972)e^x+216x^2+1404x+2079)e^{2x^*}+(-27{,}648x^2e^{3x}\\
&+(-32{,}256x-4416)e^{2x}+(2304x-7296)e^x+864)e^{x^*}+((864x^2+12{,}096x+6912)e^{4x}\\
&+5184e^{3x}-432e^{2x})x^*+(-288x^3-6480x^2+28{,}512x+10{,}368)e^{4x}+(17{,}280\\
&-12{,}096x)e^{3x}+(1584x-7752)e^{2x}+1584e^x-108)\Big]\\
&-\frac{e^{-2x^*-4x}}{16y_0}(((16x-7)e^{2x}+12e^x-1)e^{2x^*}-(16xe^{3x}+8e^{2x})e^{x^*}+2e^{4x}x^*+(7-2x)e^{4x}\\
&-4e^{3x}+e^{2x})-\frac{e^{-4x}}{16y_0}(20e^{3x}-(2x^2+28x+9)e^{2x}-12e^x+1)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\chi_> = \pi cN(T_0-2V_0)\Big\{&\varepsilon^2\Big[\frac{e^{-2x^*-2x}}{55{,}296}((10{,}756e^x-7344x-6745)e^{2x^*}+(-23{,}424e^x+17{,}280x\\
&+12{,}576)e^{x^*}+(8640e^x+3024)x^*+10{,}608e^x+3024x+10{,}764)\\
&-\frac{e^{-6x^*-2x}}{55{,}296y_0}((17{,}280x^*-17{,}280x-34{,}560)e^{5x^*}+(-1152x^{*3}+(1728x-18{,}144)x^{*2}\\
&+(34{,}560e^x+25{,}920x+36{,}720)x^*+17{,}280e^x+23{,}760x+54{,}216)e^{4x^*}+(-2304e^xx^{*3}\\
&+(-43{,}776e^x-1152)x^{*2}+(-39{,}552e^x-28{,}416)x^*-1664e^x+6912x+2624)e^{3x^*}\\
&+((216-864e^x)x^{*2}+(4140-30{,}960e^x)x^*-12{,}732e^x-432x-7593)e^{2x^*}\\
&+(1152e^xx^*-5376e^x+2016)e^{x^*}+432e^x-108)\\
&-\frac{e^{-2x^*-6x}}{13{,}824y_0}((((576x-348)e^{2x}+504e^x-54)x^{*2}+((-1152x^2-3144x+2264)e^{2x}\\
&+(-1008x-3324)e^x+108x+351)x^*+(8640x-4320)e^{3x}+(2016x+5982)e^{2x}\\
&+1044e^x-189)e^{2x^*}+(((3456-6912x)e^{3x}-5760e^x+576e^x)x^*+(3456\\
&-6912x)e^{3x}+(-2304x-4368)e^{2x}-1440e^x+216)e^{x^*}+(864-1728x)e^{3x}\\
&+(288x-1614)e^{2x}+396e^x-27)
\end{aligned}
$$

$$+\frac{e^{-6x^*-2x}}{13{,}824y_0}((4320x^*-4320x-8640)e^{5x^*}+(-216x^{*3}+(216x-4428)x^{*2}+(8640e^x$$
$$+4968x+7884)x^*+4320e^x+2700x+9342)e^{4x^*}+(-576e^xx^{*3}+(-10{,}944e^x-288)x^{*2}$$
$$+(-9760e^x-7104)x^*+96e^x+1728x+720)e^{3x^*}+((54-216e^x)x^{*2}+(1035-7740e^x)x^*$$
$$-3180e^x-108x-1899)e^{2x^*}+(288e^xx^*-1344e^x+504)e^{x^*}+108e^x-27)$$
$$+\frac{e^{-2x^*-6x}}{9216y_0}((((384x-234)e^{2x}+336e^x-36)x^{*2}+((-768x^2-2092x+1530)e^{2x}+(-672x$$
$$-2216)e^x+72x+234)x^*+(5760x-3200)e^{3x}+(1344x+3981)e^{2x}+696e^x-126)e^{2x^*}$$
$$+(((2560-4608x)e^{3x}-3840e^{2x}+384e^x)x^*+(2560-4608x)e^{3x}+(-1536x-2904)e^{2x}$$
$$\left.-960e^x+144)e^{x^*}+(640-1152x)e^{3x}+(192x-1077)e^{2x}+264e^x-18)\right]$$
$$\left.-\frac{e^{-2x^*-2x}}{16y_0}((2x^{*2}+(-4x-12)x^*+20e^x+7)e^{2x^*}+(-16e^xx^*-16e^x-8)e^{x^*}-4e^x+1)\right\}.$$

Pure radial drive

$$\chi_< = \pi cNU_0\frac{e^{-3x^*-4x}}{6912y_0}\left\{\varepsilon^2\left[(-2928e^{3x}+(2160x+1572)e^{2x}+(192x+704)e^x+6)e^{3x^*}\right.\right.$$
$$+((4320e^{3x}-(432x^2+2592x+3456)e^{2x})x^*+984e^{3x}+(144x^3+216x^2-774)e^{2x}$$
$$\left.+192e^x-3)e^{2x^*}+(576xe^{3x}+288e^{2x})e^{x^*}-12e^{4x}x^*+(12x-59)e^{4x}+24e^{3x}-6e^{2x}\right]$$
$$\left.+(6912xe^{3x}+3456e^{2x})e^{3x^*}+(-3456e^{4x}x^*+(3456x-8640)e^{4x}+6912e^{3x}-1728e^{2x})e^{2x^*}\right\},$$
$$\chi_> = -\pi cNU_0\frac{e^{-3x^*-4x}}{2304y_0}\left\{\varepsilon^2\left[e^{4x^*}+(-64e^xx^*+976e^{3x}-(524+1152+720x)e^{2x}+64e^x-2)e^{3x^*}\right.\right.$$
$$+(-48e^{2x}x^{*3}+(144x+72)e^{2x}x^{*2}+((720x+432)e^{2x}-1440e^{3x})x^*-328e^{3x}$$
$$\left.+(720x+618)e^{2x}-64e^x+1)e^{2x^*}+(-192e^{3x}x^*-640e^{3x}-96e^{2x})e^{x^*}-8e^{3x}+2e^{2x}\right]$$
$$\left.+576e^{2x}e^{4x^*}+2304e^{3x}(1-x^*)e^{3x^*}+(-2304e^{3x}+576e^{2x})e^{2x^*}\right\}. \qquad (26)$$

ILLUSTRATIONS

Here we examine some approximate two dimensional hydrodynamical solutions. As you might expect end effects produce axial flow taper as well as reduce the peak axial flow and simultaneously generate non-zero radial flow. Flow blockage associated with the impermeable end-caps is obvious.

Axial mass velocity profiles to $O(\varepsilon^4)$ for all the source/sink drives U, V, W, T are plotted in Figures 1 and 2 for $\varepsilon = 0, 1, 2, 3$. Suppose $c = -1, N = 1, y_0 = 40$ and the unit source/sink strengths are chosen to produce downflow at the wall, that is $U_0 = -1$, $W_0 = T_0 = 1$ and $V_0 = 0$. Let $x^* \to \infty$ for simplicity. Since in this limit the pure axial and pure radial drives differ only by a constant we include graphs of W only. To $O(\varepsilon^2)$ the axial (and of course radial) drive solution fails for $1 < \varepsilon \leqslant 2$ whereas the heat and drag drive solutions have a larger region of validity, failing for $2 < \varepsilon \leqslant 3$. Indeed, while they may be poor approximations for any $\varepsilon > 1$ they fail to even represent the proper structure outside these regions of validity. Perhaps this is an alternating series. Obviously, this domain is extended when higher order terms, say $O(\varepsilon^4)$, are included. Owing to the progressive complexity, it does not seem practical to go so far as to compute the complete $O(\varepsilon^6)$ terms in the same manner. It might however be possible to calculate the asymptotic solutions ($x^* \to \infty$). Apparently, the best way to calculate the flow for $\varepsilon \gg 1$ is via the short bowl theory which results in a different set of equations and perturbation techniques.

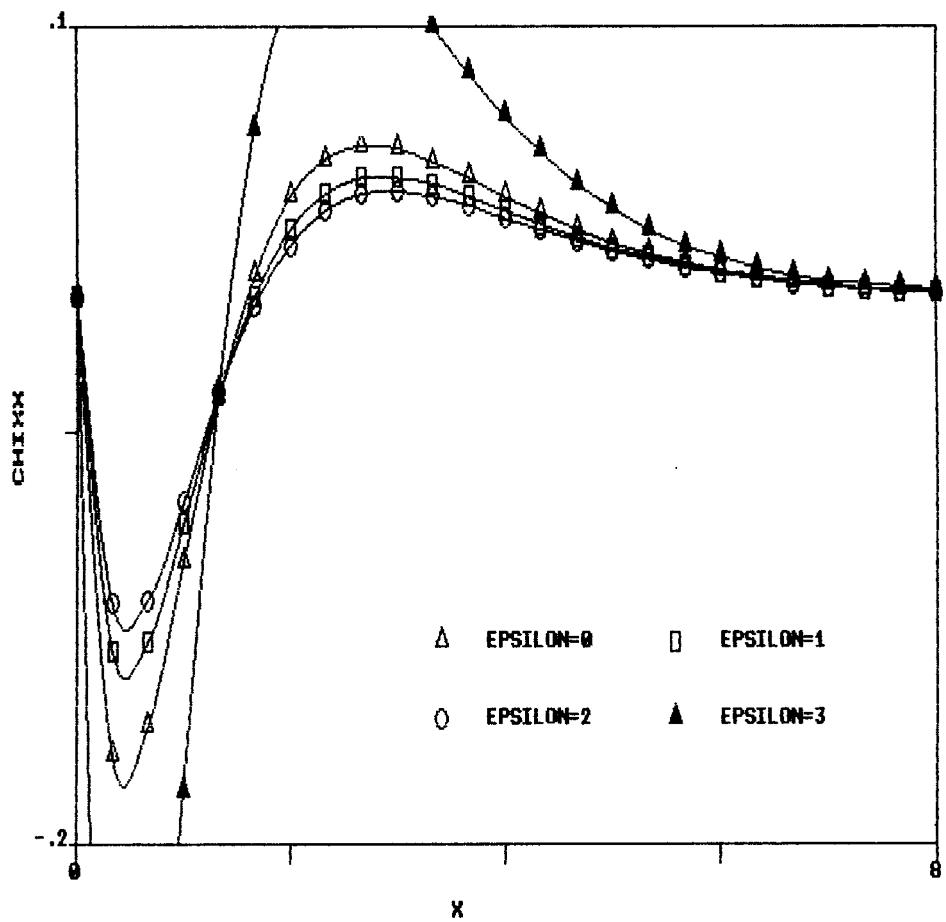

Figure 1. χ''_∞ versus x for asymptotic pure axial drive and $\varepsilon = 0, 1, 2, 3$

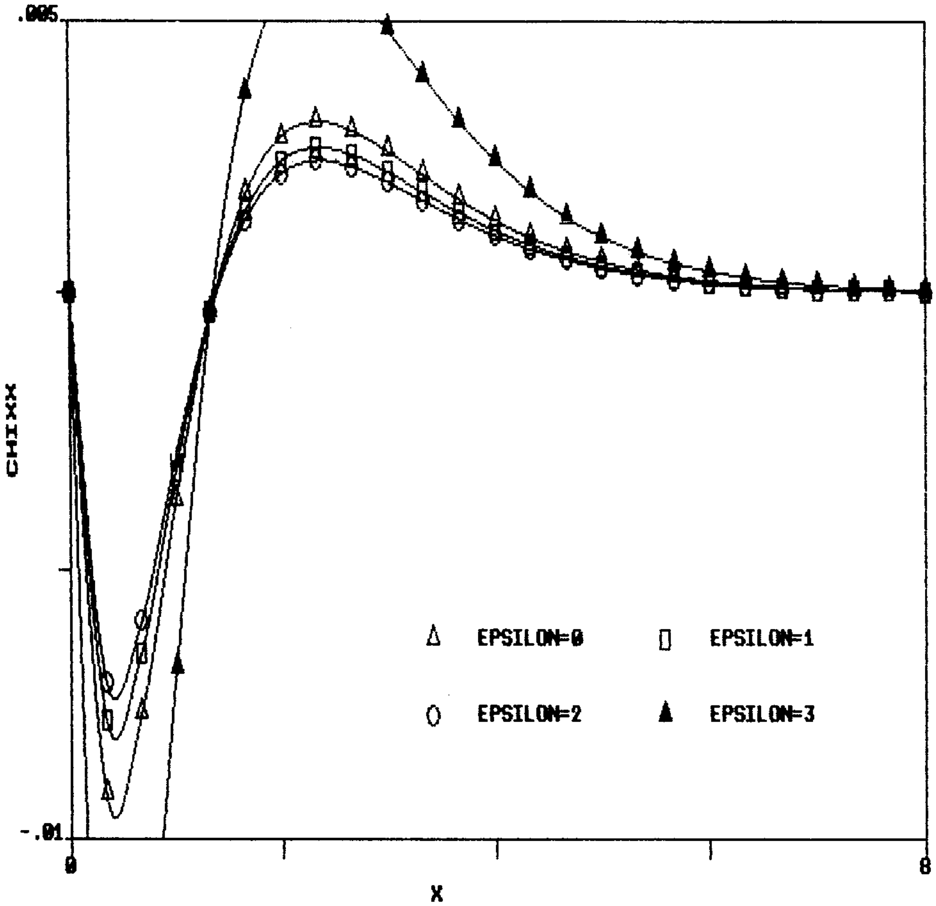

Figure 2. χ''_∞ versus x for asymptotic pure heat and drag drive and $\varepsilon = 0, 1, 2, 3$

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NOMENCLATURE

A	Stratification parameter $\left[\dfrac{MV_w^2}{2RT_0}\right]^{1/2}$
B	$\dfrac{ReS^{1/2}}{4A^6}$

c	$\dfrac{-B^2A^2}{2ReS}$
E	$BN\pi/y_0$
$\tilde{F}(x,y)$, $F(x)$	Inhomogeneities
G	Green's function
$L\tilde{\chi}$	$L_6\tilde{\chi}+B^2\tilde{\chi}_{yy}$
$L_6\tilde{\chi}$	$[e^x(e^x\tilde{\chi}_{xx})_{xx}]_{xx}$
M	Molecular weight, or order of expansion
N	Number of half-cycles
Pr	Prandtl number
R	Universal gas constant
Re	Reynolds number
S	$1+\dfrac{(\gamma-1)}{2\gamma}PrA^2$
T_0	Reference temperature
$\tilde{U},\tilde{V},\tilde{W},\tilde{T}$	Sources and sinks
U_0, V_0, W_0, T_0	Source/sink strengths
U	Heaviside unit step function
V_w	Wall velocity
x	Scale heights variable
x^*	Source point
y	Dimensionless axial co-ordinate
y_0	Dimensionless rotor length
γ	Specific heat ratio
δ	Dirac delta
ε	Small expansion parameter, $BN\pi/y_0$
$\bar{\theta}_y$	Sidewall temperature gradient
ξ	Dummy integration variable
$\rho_0\tilde{w}$	Dimensionless axial velocity
$\tilde{\chi}$	Master potential
$\tilde{\psi}$	Stream function
$\sim$	Multivariate function

APPENDIX—$O(\varepsilon^4)$ SOLUTIONS

In general,

$$\chi_M(x)=\int_0^{-\infty}G(x,\xi)\chi_{M-2}(\xi)\mathrm{d}\xi,\ M=2,4,\ldots \tag{27}$$

Thus a regular expansion can be derived to arbitrary order M, in theory, 'simply' by one-dimensional quadratures. And it is limited only by the radius of convergence of our asymptotic series. This problem is well suited to Van Dyke's technique[8] as well. Specifically, for $M=4$ we have the following.

Pure axial drive

$$\chi_{4_<}=-cW_0\frac{e^{-5x^*-6x}}{248{,}832{,}000}[e^{x^*}(e^{x^*}(e^{x^*}(e^{x^*}(e^{x}(10e^{x}(39{,}075e^{x^*}-4{,}840{,}200e^{3x}x^*+3{,}304{,}800xe^{2x}x^*$$

$$+ 3{,}035{,}250\mathrm{e}^{2x}x^* + 864{,}000\mathrm{e}^{x}x^* - 5400x^2x^* - 55{,}800xx^* + 307{,}874\mathrm{e}^{3x} - 269{,}550x\mathrm{e}^{2x}$$
$$- 115{,}005\mathrm{e}^{2x} - 235{,}200\mathrm{e}^{x}) + (5\mathrm{e}^{2x}(3{,}573{,}744\mathrm{e}^{2x} - (2{,}420{,}100x + 2{,}266{,}163)\mathrm{e}^{x} - 585{,}600)$$
$$+ 120x(1125\mathrm{e}^{x} + 16) + 10{,}048)\mathrm{e}^{x^*} - 250\mathrm{e}^{x}(5733x^* - 72x^3 - 684x^2 - 2781x - 4258))$$
$$+ 15\mathrm{e}^{x^*} + 3840\mathrm{e}^{x}) - 15\mathrm{e}^{x^*} - 18{,}000\mathrm{e}^{2x}(6(244\mathrm{e}^{x} - 180x - 131)\mathrm{e}^{2x} - 32(3x + 11)\mathrm{e}^{x} - 3))$$
$$+ 1500\mathrm{e}^{2x}(2\mathrm{e}^{2x}(72(10\mathrm{e}^{x} - (x + 2)(x + 4))x^* + 1724\mathrm{e}^{x} + 3(8x^3 - 40x^2 - 312x - 459))$$
$$+ 64\mathrm{e}^{x} - 1)) + 24{,}000(2x\mathrm{e}^{x} + 1)\mathrm{e}^{4x}) - 9\mathrm{e}^{4x}(2\mathrm{e}^{2x}(15x^* - 15x + 91) - 60\mathrm{e}^{x} + 15)],$$

$$\chi_{4_>} = -cW_0\frac{\mathrm{e}^{-5x^*-6x}}{49{,}766{,}400}[\mathrm{e}^{x^*}(\mathrm{e}^{x^*}(\mathrm{e}^{x^*}(\mathrm{e}^{x^*}(\mathrm{e}^{x}(((3{,}573{,}744\mathrm{e}^{x} - 2{,}420{,}100x - 2{,}266{,}163)\mathrm{e}^{2x} - 585{,}600\mathrm{e}^{x}$$
$$+ 27{,}000x + 78{,}150)\mathrm{e}^{x^*+x} + 384(x^* - 2)\mathrm{e}^{x^*} + 2\mathrm{e}^{x}(1800x^{*3} - 5400xx^{*2} - 17{,}100x^{*2}$$
$$- 4{,}840{,}200\mathrm{e}^{3x}x^* + 3{,}304{,}800x\mathrm{e}^{2x}x^* + 3{,}035{,}250\mathrm{e}^{2x}x^* + 864{,}000\mathrm{e}^{x}x^* - 21{,}600xx^*$$
$$+ 307{,}874\mathrm{e}^{3x} - 269{,}550x\mathrm{e}^{2x} - 115{,}005\mathrm{e}^{2x} - 235{,}200\mathrm{e}^{x}) - 450\mathrm{e}^{x}(268x^* + 60x + 197))$$
$$+ 3\mathrm{e}^{x^*} + 768\mathrm{e}^{x}) - 3\mathrm{e}^{x^*} + 10{,}800\mathrm{e}^{2x}(32\mathrm{e}^{x}(x^* + 3) - 2(244\mathrm{e}^{x} - 180x - 131)\mathrm{e}^{2x} + 1))$$
$$+ 100\mathrm{e}^{2x}(2\mathrm{e}^{2x}(36x^*(2x^*(x^* - 3x + 5) + 60\mathrm{e}^{x} - 56x + 3) + 5172\mathrm{e}^{x} - 4644x - 1691) + 192\mathrm{e}^{x}$$
$$- 3)) + 1600\mathrm{e}^{4x}(\mathrm{e}^{x}(6x^* + 29) + 3)) + 27(4\mathrm{e}^{x} - 1)\mathrm{e}^{4x}] \tag{28}$$

$$\chi''_\infty = \frac{-cW_0}{12{,}441{,}600}\{\varepsilon^4[893{,}436\mathrm{e}^{-x} - (2{,}420{,}100x - 153{,}937)\mathrm{e}^{-2x} - 1{,}317{,}600\mathrm{e}^{-3x} + (108{,}000x$$
$$+ 258{,}600)\mathrm{e}^{-4x} + (2400x + 11{,}600)\mathrm{e}^{-5x} + 27\mathrm{e}^{-6x}] + \varepsilon^2[-5{,}270{,}400\mathrm{e}^{-x} + (15{,}552{,}000x$$
$$- 4{,}233{,}600)\mathrm{e}^{-2x} + (3{,}110{,}400x + 9{,}331{,}200)\mathrm{e}^{-3x} + 172{,}800\mathrm{e}^{-4x}] + (12{,}441{,}600x$$
$$- 24{,}883{,}200)\mathrm{e}^{-x} + 24{,}883{,}200\mathrm{e}^{-2x}\}. \tag{29}$$

Pure heat and drag drive

$$\chi_{4_<} = -\pi cN(T_0 - 2V_0)\frac{\mathrm{e}^{-6x^*-6x}}{59{,}719{,}680{,}000y_0}[\mathrm{e}^{x^*}(\mathrm{e}^{x^*}(\mathrm{e}^{x^*}(\mathrm{e}^{x^*}(\mathrm{e}^{x^*}(75\mathrm{e}^{x}(\mathrm{e}^{x}(566{,}425\mathrm{e}^{x^*} - 16(3573{,}744\mathrm{e}^{x}$$
$$- 2{,}420{,}100x - 2{,}266{,}163)\mathrm{e}^{2x} + 9{,}369{,}600\mathrm{e}^{x}) + 7680\mathrm{e}^{x^*} - 2400(180x + 521)\mathrm{e}^{x})$$
$$+ (3(664{,}832{,}112\mathrm{e}^{5x} - 449{,}572{,}500x\mathrm{e}^{4x} - 422{,}333{,}515\mathrm{e}^{4x} - 107{,}560{,}000\mathrm{e}^{3x} + 4{,}590{,}000x\mathrm{e}^{2x}$$
$$- 300x^2 - 3840x) - 36{,}961)\mathrm{e}^{x^*} - 15{,}360(30x + 157)\mathrm{e}^{x}) - 3600\mathrm{e}^{x^*} + 150\mathrm{e}^{2x}(450(2(10{,}756\mathrm{e}^{x}$$
$$- 7344x - 6745)\mathrm{e}^{2x} - 3840\mathrm{e}^{x} + 24x^2 + 248x + 637)x^* + 4(3{,}476{,}213\mathrm{e}^{x} - 2{,}343{,}825x$$
$$- 2{,}218{,}935)\mathrm{e}^{2x} - 2{,}121{,}600\mathrm{e}^{x} - 25(144x^3 + 720x^2 - 1{,}134x - 8683)) - 115{,}200\mathrm{e}^{x}) + 450\mathrm{e}^{x^*}$$
$$+ 160{,}000\mathrm{e}^{2x}(6(244\mathrm{e}^{x} - 180x - 131)\mathrm{e}^{2x} - 32(3x + 11)\mathrm{e}^{x} - 3)) - 5625\mathrm{e}^{2x}(2\mathrm{e}^{2x}(72(10\mathrm{e}^{x}$$
$$- (x + 2)(x + 4))x^* + 2264\mathrm{e}^{x} + 3(8x^3 - 58x^2 - 420x - 603)) + 64\mathrm{e}^{x} - 1)) - 46{,}080(2x\mathrm{e}^{x}$$
$$+ 1)\mathrm{e}^{4x}) + 10\mathrm{e}^{4x}(\mathrm{e}^{2x}(30x^* - 30x + 197) - 60\mathrm{e}^{x} + 15)],$$

$$\chi_{4_>} = -\pi c N(T_0 - 2V_0)\frac{e^{-6x^*-6x}}{19{,}906{,}560{,}000 y_0}[e^{x^*}(e^{x^*}(e^{x^*}(e^{x^*}(e^{x^*}(25e^x(e^x(566{,}425e^{x^*} - 16(3{,}573{,}744e^x$$
$$- 2{,}420{,}100x - 2{,}266{,}163)e^{2x} + 9{,}369{,}600e^x) + 7680e^{x^*} - 2400(180x + 521)e^x + (60x^*(5x^*$$
$$- 10x - 64) + e^{2x}((664{,}832{,}112e^x - 449{,}572{,}500x - 422{,}333{,}515)e^{2x} - 107{,}560{,}000e^x$$
$$+ 4{,}590{,}000x) + 1050)e^{x^*} - 153{,}600e^x(x^* + 1)) - 1200e^{x^*} - 100e^{2x}(450x^*(4x^*(x^* - 3x - 5)$$
$$-(10{,}756e^x - 7344x - 6745)e^{2x} + 1920e^x - 84x - 212) - 2(3{,}476{,}213e^x$$
$$- 2{,}343{,}825x - 2{,}218{,}935)e^{2x} + 1{,}060{,}800e^x - 62{,}100x - 172{,}575) - 38{,}400e^x) + 150e^{x^*}$$
$$- 160{,}000e^{2x}(32e^x(x^* + 4) - 2(244e^x - 180x - 131)e^{2x} + 1)) - 625e^{2x}(e^{2x}(36x^*(x^*(4x^*$$
$$- 12x + 29) + 120e^x - 130x + 45) + 13{,}584e^x - 12{,}636x - 2545) + 192e^x - 3))$$
$$- 1024e^{4x}(e^x(30x^* + 163) + 15)) - 50(4e^x - 1)e^{4x}]. \tag{30}$$

$$\chi''_\infty = \frac{-\pi c N(T_0 - 2V_0)}{4{,}976{,}640{,}000 y_0}\{\varepsilon^4[166{,}208{,}028e^{-x} - (449{,}572{,}500x - 27{,}238{,}985)e^{-2x} - 242{,}010{,}000e^{-3x}$$
$$+ (18{,}360{,}000x + 47{,}462{,}500)e^{-4x} + 1{,}200{,}000e^{-5x} - (2700x^2 + 32{,}760x + 99{,}513)e^{-6x}]$$
$$+ \varepsilon^2[-968{,}040{,}000e^{-x} + (2{,}643{,}840{,}000x - 215{,}640{,}000)e^{-2x} + 1{,}555{,}200{,}000e^{-3x}$$
$$- (17{,}280{,}000x^2 + 161{,}280{,}000x + 371{,}520{,}000)e^{-4x}] + 6{,}220{,}800{,}000e^{-x}$$
$$- (2{,}488{,}320{,}000x^2 + 9{,}953{,}280{,}000x + 6{,}220{,}800{,}000)e^{-2x}\}. \tag{31}$$

Pure radial drive

$$\chi_{4_<} = \pi c N U_0 \frac{e^{-5x^*-6x}}{497{,}664{,}000 y_0}[(e^{x^*}(e^{x^*}(e^{x^*}(e^{x^*}(e^x(20e^x(39{,}075e^{x^*} - 2{,}420{,}100e^{3x}x^* + 1{,}652{,}400xe^{2x}x^*$$
$$+ 1{,}517{,}625e^{2x}x^* + 432{,}000e^x x^* - 2700x^2x^* - 27{,}900xx^* - 1{,}056{,}113e^{3x} + 691{,}425xe^{2x}$$
$$+ 701{,}310e^{2x} + 98{,}400e^x) + 2(5e^{2x}(3{,}573{,}744e^{2x} - (2{,}420{,}100x + 2{,}266{,}163)e^x - 585{,}600)$$
$$+ 120x(1125e^x + 16) + 10{,}048)e^{x^*} - 125e^x(11{,}466x^* - 144x^3 - 1152x^2 - 3330x - 2783))$$
$$+ 30e^{x^*} + 3840e^x) - 15e^{x^*} - 12{,}000e^{2x}(6(244e^x - 180x - 131)e^{2x} - 32(3x + 11)e^x - 3))$$
$$+ 750e^{2x}(2e^{2x}(72(10e^x - (x + 2)(x + 4))x^* + 1904e^x + 3(8x^3 - 46x^2 - 348x - 507))$$
$$+ 64e^x - 1)) + 9600(2xe^x + 1)e^{4x}) - 3e^{4x}(e^{2x}(30x^* - 30x + 187) - 60e^x + 15)],$$

$$\chi_{4_>} = -\pi c N U_0 \frac{e^{-5x^*-6x}}{99{,}532{,}800 y_0}[(e^{x^*}(e^{x^*}(e^{x^*}(e^{x^*}(e^{x^*}(3e^{x^*} - 768e^x(x^* - 1) - 2e^{2x}((3{,}573{,}744e^x$$
$$- 2{,}420{,}100x - 2{,}266{,}163)e^{2x} - 585{,}600e^x + 27{,}000x + 78{,}150)) - 6e^{x^*} - 2e^{2x}(450x^*(4x^*(x^*$$
$$- 3x - 8) - (10{,}756e^x - 7344x - 6745)e^{2x} + 1920e^x - 60x - 166) - 2(1{,}056{,}113e^x - 691{,}425x$$
$$- 701{,}310)e^{2x} + 196{,}800e^x - 27{,}000x - 81{,}675) - 768e^x) + 3e^{x^*} - 2400e^{2x}(32e^x(3x^* + 10)$$
$$- 6(244e^x - 180x - 131)e^{2x} + 3)) - 25e^{2x}(e^{2x}(36x^*(2x^*(4x^* - 12x + 23) + 240e^x - 236x$$

$$+35)+22{,}848e^{x}-20{,}700x-6449)+384e^{x}-6))-128e^{4x}(e^{x}(30x^{*}+151)+15))$$
$$-9(4e^{x}-1)e^{4x}]. \tag{32}$$

$$\chi''_{\infty}=\frac{\pi cNU_0}{12{,}441{,}600y_0}\{\varepsilon^4[893{,}436e^{-x}+(-2{,}420{,}100x+153{,}937)e^{-2x}-1{,}317{,}600e^{-3x}+(108{,}000x$$
$$+258{,}600)e^{-4x}+(2400x+11{,}600)e^{-5x}+27e^{-6x}]+\varepsilon^2[-5{,}270{,}400e^{-x}+(15{,}552{,}000x$$
$$-4{,}233{,}600)e^{-2x}+(3{,}110{,}400x+9{,}331{,}200)e^{-3x}+172{,}800e^{-4x}]+(12{,}441{,}600x$$
$$-24{,}883{,}200)e^{-x}+24{,}883{,}200e^{-2x}\}. \tag{33}$$

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